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AN EFFECTIVE BOUNDARY ELEMENT APPROACH FOR HIGHER ORDER SINGULARITIES

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Abstract—The paper presents a new boundary element approach which introduces a "source line". It is very effective for higher order singularity. The scheme will be discussed in some detail with plane elasticity. Numerical results for meshes with unequal boundary elements are reported. Higher precision than the general boundary element method is obtained for both deflection and force.

1. INTRODUCTION

Although applications of the boundary element method (BEM) are already well developed in various fields, it is worth recalling that in some very important cases non-integrable singularities are encountered which could affect the numerical results. As we know, the boundary integral equation of plane elasticity problems contains the singularity factor $1/r$, where r is the distance between the source point and the field point. For integrals corresponding to the singular elements, generally, the higher order integration rules should be used (Brebbia *et al.*, 1984). The first author has discussed the precision of numerical results in terms of the Gauss integration order (Lei Xiao-yan and Huang Mao-kuang, 1985). With this approach it is necessary to select the number of Gaussian points required to give sufficient precision in the integration (Lin Jun *et al.*, 1985). On the other hand, a rigid body movement may be defined to deal with the non-integrable singularities (Brebbia and Walker, 1979). For the analysis of thin plate bending by BEM, the singularity factor $1/r^2$ is encountered (Bezine and Gamby, 1978; Stern, 1979). The treatment of the non-integrable terms is very important with free edges. Stefano (1984) obtained a weak formulation of BEM deduced from distribution theory, which took the constant element interpolation at boundary elements, and applied it to elasticity problems. However, this work did not discuss the calculation of the higher order singularity of a boundary integral equation and the stability of numerical solutions. Polizzgtto (1991) presented a boundary element formulation from a boundary min-max principle without a test example. Dyka and Millwater (1987) use shape functions for quadratic elements to eliminate the strong $1/r$ singularities.

In this paper, a new boundary integral equation with application to plane elasticity problems, derived using a "source line", is presented. The linear interpolation for variables at boundary elements is considered. We shall discuss in some detail the reduction of higher order singularity in the boundary integral equations. The transformation of double integration into linear integration and the singular coefficient for non-smooth boundary elements is described. This approach gives higher precision in the numerical examples than the general BEM, especially for the meshes of unequal length boundary segments which is discussed less in previous works on BEM.

2. THE BOUNDARY INTEGRAL EQUATIONS DERIVED BY "SOURCE LINE"

Consider for simplicity two-dimensional elasticity problems. The boundary integral equation can be written as

$$C_{ij}u_j(P)+\int_{\Gamma_Q} u_j(Q)q_{ij}^*(P,Q)\,\mathrm{d}\Gamma_Q = \int_{\Gamma_Q} q_j(Q)u_{ij}^*(P,Q)\,\mathrm{d}\Gamma_Q, \tag{1}$$

where P and Q are the so-called source point and field point. The boundary Γ is divided into N boundary elements, of which Γ_{pm} is the mth element and considered as the "source line". Taking $g(P)$ as the weighting function selected according to the singularity of $u^*(P, Q)$ and $q^*(P, Q)$, which will be discussed later, and multiplying eqn (1) by the function $g(P)$, then integrating the equation along Γ_{pm}, one obtains

$$C_{ij}\int_{\Gamma_{pm}} u_j(P)g(P)\,\mathrm{d}\Gamma_{pm} = \int_{\Gamma_{pm}}\int_{\Gamma_Q} g(P)u_{ij}^*(P,Q)q_j(Q)\,\mathrm{d}\Gamma_Q\,\mathrm{d}\Gamma_{pm} - \int_{\Gamma_{pm}}\int_{\Gamma_Q} g(P)q_{ij}^*(P,Q)u_j(Q)\,\mathrm{d}\Gamma_Q\,\mathrm{d}\Gamma_{pm}. \tag{2}$$

Discretizing eqn (2) and considering the nth element Γ_{Qn}, we examine the integrals that relate the "source line" Γ_{pm} to the boundary element Γ_{Qn}. The integral including $u_{ij}^*(P, Q)$ is of the form

$$G_{mn} = \int_{\Gamma_{pm}}\int_{\Gamma_{Qn}} g(P)u_{ij}^*(P,Q)q_j(Q)\,\mathrm{d}\Gamma_{pm}\,\mathrm{d}\Gamma_{Qn}. \tag{3}$$

For the linear elements the Cartesian coordinates of points located within each element Γ_{Qn} are expressed in terms of interpolation functions N_k ($k = 1, 2$), and the coordinates of the element, boundary displacements and tractions are approximated over each segment through a linear interpolation function:

$$F = F_1N_1+F_2N_2, \tag{4}$$

$$N_1 = (1-\xi)/2, \quad N_2 = (1+\xi)/2. \tag{5}$$

The variables and coordinates within boundary elements corresponding to the source point P and field point Q, refer to the local coordinates of η and ξ respectively. Therefore eqn (3) is given by

$$G_{mn} = G_{mn1}q_{n1}+G_{mn2}q_{n2} \tag{6}$$

in which

$$G_{mnk} = \tfrac{1}{4}L_{pm}L_{Qn}\int_{-1}^{1}\int_{-1}^{1} g(\eta)u_{ij}^*(\xi,\eta)N_k(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{7}$$

where $k = 1, 2$, and q_{nk} stands for the nodal tractions of the nth boundary element. The G_{mnk} are influence coefficients defining the interaction between the line Γ_{Qn} under consideration and the "source line" Γ_{pm}. The lengths of Γ_{Qn} and Γ_{pm} are L_{Qn} and L_{pm}, respectively.

3. TRANSFORMATION OF DOUBLE INTEGRATION

The term G_{mnk} defined by eqn (7) is a double integration. It can be transformed into a linear integration by some linear transformation as follows. In eqns (4) and (5), the coordinates of the field and source may be written in terms of their local coordinates ξ and η, respectively:

$$X_Q = \tfrac{1}{2}(X_{Q2}+X_{Q1})+\tfrac{1}{2}(X_{Q2}-X_{Q1})\xi,$$
$$Y_Q = \tfrac{1}{2}(Y_{Q2}+Y_{Q1})+\tfrac{1}{2}(Y_{Q2}-Y_{Q1})\xi, \tag{8}$$

$$X_p = \tfrac{1}{2}(X_{p2}+X_{p1})+\tfrac{1}{2}(X_{p2}-X_{p1})\eta,$$
$$Y_p = \tfrac{1}{2}(Y_{p2}+Y_{p1})+\tfrac{1}{2}(Y_{p2}-Y_{p1})\eta, \tag{9}$$

where the subscripts 1 and 2 indicate the nodes of the element. Let us take the transformation

$$X_Q-X_p = C_1+C_2\xi-C_3\eta = r\cos\theta,$$
$$Y_Q-Y_p = b_1+b_2\xi-b_3\eta = r\sin\theta, \tag{10}$$

in which the coefficients C_i and b_i ($i = 1, 2, 3$) can be deduced from eqns (8) and (9) which relate to the nodal coordinates of boundary elements. The Jacobian is given by

$$J = \frac{D(\xi,\eta)}{D(r,\theta)} = \frac{1}{a}r\,\mathrm{d}r\,\mathrm{d}\theta, \tag{11}$$

$$a = b_2C_3-C_2b_3. \tag{12}$$

When $a \neq 0$, one obtains from eqn (10)

$$\xi = \frac{1}{a}[r(C_3\sin\theta-b_3\cos\theta)+b_3C_1-b_1C_3],$$

$$\eta = \frac{1}{a}[r(C_2\sin\theta-b_2\cos\theta)+b_2C_1-b_1C_2]. \tag{13}$$

Substituting eqns (11) and (13) into (7), the integral region is constructed by the local coordinates $\xi = \mp 1$ and $\eta = \mp 1$ (see Fig. 1), of which $ABCD$ is expressed by $\Gamma_{(r'\theta')}$. Equation (7) becomes

$$G_{mnk} = \frac{1}{4a}L_{pm}L_{Qn}\int\int_{\Gamma_{(r'\theta')}} g(r,\theta)u_{ij}^*(r,\theta)N_k(r,\theta)r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{14}$$

In the cylindrical coordinate system (r, θ), the boundary lines (see Fig. 1) are defined by eqn (13) when considering $\xi = \mp 1$ and $\eta = \mp 1$, respectively. The integral along r can be evaluated analytically, so eqn (14) can be transformed into linear integration.

When $a = 0$, the boundary elements Γ_{pm} and Γ_{Qn} are parallel. For $m \neq n$ and $C_2 \neq 0$, eqn (10) leads to

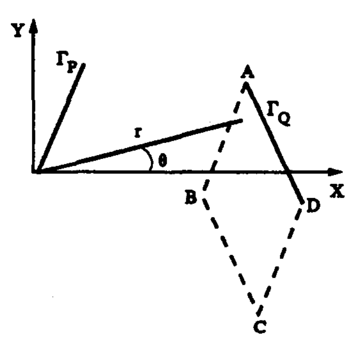

Fig. 1.

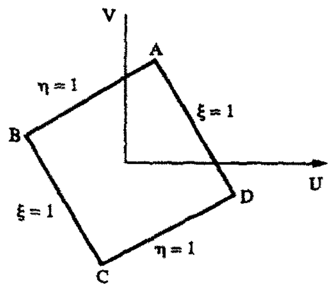

Fig. 2.

$$X_Q - X_p = C_1 + C_2\xi - C_3\eta,$$
$$Y_Q - Y_p = b_1 + k(C_2\xi - C_3\eta), \tag{15}$$

where $k = b_2/C_2 = b_3/C_3$, assume

$$U = C_2\xi - C_3\eta,$$
$$V = C_3\xi + C_2\eta, \tag{16}$$

then

$$\xi = \frac{1}{C_2^2 + C_3^2}(C_3V + C_2U),$$
$$\eta = \frac{1}{C_2^2 + C_3^2}(C_2V - C_3U). \tag{17}$$

The Jacobian is $J = (D(\xi, \eta)/D(U, V)) = 1/A$, in which $A = C_2^2 + C_3^2$. $\xi = \mp 1$ and $\eta = \mp 1$ map the integral region of Cartesian coordinates (U, V) (see Fig. 2) expressed by $\Gamma_{(U'V')}$. Substituting eqns (16) and (17) into (7), we have

$$G_{mnk} = \frac{1}{4A} L_{pm} L_{Qn} \int\int_{\Gamma_{(U'V')}} g(U, V) u_{ij}^*(U) N_k(U, V)\, dU\, dV. \tag{18}$$

This equation is similar to eqn (14). Considering the boundary lines defined by eqn (17) when $\xi = \mp 1$ and $\eta = \mp 1$ (see Fig. 2), respectively. Equation (18) can thus be transformed into linear integration with the integration in the V or U direction calculated analytically.

4. ANALYTICAL SINGULARITY INTEGRATION

When $a = 0$ and $m = n$, eqns (10) and (7) become

$$X_Q - X_p = \tfrac{1}{2}(X_2 - X_1)(\xi - \eta),$$
$$Y_Q - Y_p = \tfrac{1}{2}(Y_2 - Y_1)(\xi - \eta), \tag{19}$$

$$G_{nnk} = \tfrac{1}{4} L_n^2 \int_{-1}^{1} \int_{-1}^{1} g(\eta) u_{ij}^*(\xi - \eta) N_k(\xi)\, d\xi\, d\eta. \tag{20}$$

For boundary integral equation (1) of the plane elasticity problem, the fundamental solutions $u_{ij}^*(P, Q)$ and $q_{ij}^*(P, Q)$ have singularities $\ln r$ and $1/r$, respectively. Allowing the weight $g(P) = 1$, the integrals G_{nnk} which contain the term $u_{ij}^*(P, Q)$ and also the integrals in (2) which contain $q^*_{ij}(P, Q)$ can be evaluated analytically. In fact, when $g(P) = 1$, the double integration of eqn (20) reduces the singularity order by one compared to the general BEM. To perform the integration of (20) along with other influence coefficients, the following identities are presented:

$$\int_{-1}^{1}\int_{-1}^{1} \ln(\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = 4\ln 2-6, \quad \int_{-1}^{1}\int_{-1}^{1} \xi\ln(\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = 0,$$

$$\int_{-1}^{1}\int_{-1}^{1} \frac{1}{\xi-\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad \int_{-1}^{1}\int_{-1}^{1} \frac{\xi}{\xi-\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta = 2. \tag{21}$$

The selection of the weight $g(P)$ is very important. For the integrals corresponding to the singular elements which have the singular factor $1/r^n$ $(n > 0)$ in the boundary integral equation, the choice of $g(P) = (1-\eta^2)^{n-1}$ allows the integrals to be analytically calculated.

In boundary integral equation (1), the coefficient C_{ij} is based on the geometry of the adjacent elements. If the boundary is smooth at the nodes or constant element interpolation of which the node is taken to be in the middle of each segment, the C_{ij} coefficient is identically

$$C_{ij} = \begin{cases} 1/2 & i = j, \\ 0 & i \neq j. \end{cases} \tag{22}$$

When the curve is discontinuous at boundary point P, the coefficient C_{ij} is based on the angle between two sides of P for the general boundary element method. What is the coefficient C_{ij} for the present approach? In eqn (2), when the variables vary linearly over the element Γ_{pm}, let us assume that the "source line" Γ_{pm} is made of K subelements. For every one such as Γ^k_{pm}, a boundary integral equation similar to eqn (2) exists in which the constant element is defined over the segment Γ^k_{pm}, it gives the same C_{ij} as eqn (22). Letting K go to infinity and summing up the K equations leads to eqn (2), so the C_{ij} coefficient in eqn (2) is given by eqn (22). Thus this approach is suited to both smooth and non-smooth boundaries for linearly interpolate elements.

5. NUMERICAL RESULTS

Since the singularity order is reduced by one when $g(P) = 1$, compared with the general BEM, and the singular elements are evaluated analytically, we employ four integration points of Gaussian Quadrature for the linear integration indicated in eqns (14) and (18). In order to show the applicability and effectiveness of the numerical procedures described in this paper, three test examples are presented. In this section, the integration orders for general BEM are that 16 Gauss integration points are employed for singular elements and four points are used in other cases. The values of displacement listed in this section are multiplied by Young's modulus, $v = 0.3$.

Example (1). Uniformly loaded plane stress problems

A square plate is stretched by a uniformly distributed load over two opposite sides (see Fig. 3). This test emphasises the meshes with unequal elements. The boundary of the plate

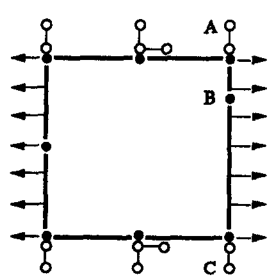

Fig. 3. A square plate is stretched by a uniformly distributed load over two opposite sides.

Table 1. Uniformly loaded plane stress problems

	General BEM		Present BEM		Exact	
L_{AB}/L_{BC}	$u_x(A)$	$q_y(A)$	$u_x(A)$	$q_y(A)$	$u_x(A)$	$q_y(A)$
1	0.9085	0.3003	0.9100	0.2996	0.9100	0.3000
3/7	0.9191	0.2621	0.9097	0.3040	0.9100	0.3000
1/9	0.9370	0.2123	0.9101	0.3035	0.9100	0.3000

Table 2. Bending in-plane of rectangular plate

L_{AB}/L_{CD}	General BEM	Present BEM	Exact
1	0.9807	1.001	1.000
1.5	1.010	1.004	1.000
2.0	1.208	1.014	1.000
5.0	0.2471	0.9920	1.000

is discretized into eight elements. Table 1 gives the displacement and traction at node A, in which L_{AB}/L_{BC} refers to the ratio of lengths AB and BC of line AC. When point B is close to node A so that $L_{AB}/L_{BC} = 1/9$, it is seen that the present solution yields satisfactory results, but that of the general BEM is in considerable error!

Example (2). Bending in-plane of a rectangular plate

A rectangular plate subjected to a bending in-plane has one side clamped and the others free (see Fig. 4). The boundary is divided into eight elements. Nodes B and D indicate the middle of the sides of the plate. Table 2 gives the traction q_x at node A, in which L_{AB}/L_{CD} stands for the ratio of adjacent sides of the plate.

Example (3). A circular plate with uniformly distributed out-normal load

Figure 5 indicates a circular plate with uniformly distributed out-normal load. The number of linear elements of the total boundary is varied from 12 to 32. The displacements in the radius direction are listed in Table 3.

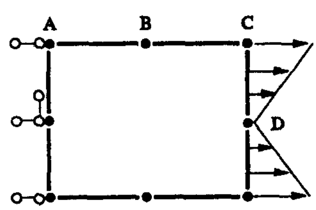

Fig. 4. A rectangular plate subjected to a bending in-plane.

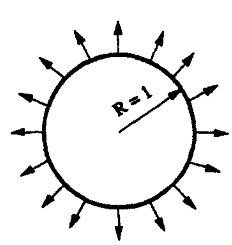

Fig. 5. A circular plate with uniformly distributed out-normal load.

Table 3. Displacements along the radius ($R = 1$) for the varied grid in Fig. 5

Number of nodes	General BEM	Present BEM	Exact
12	0.6662	0.6912	0.700
16	0.6742	0.6952	0.700
20	0.6793	0.6969	0.700
24	0.6828	0.6979	0.700
28	0.6853	0.6985	0.700
32	0.6872	0.6980	0.700

6. DISCUSSION AND CONCLUSION

(1) An efficient boundary integral equation derived by "source line" has been investigated in detail. This approach seems very useful for the boundary integral equation with higher order singularities. The method presented in this paper allows the selecting of a weighting function $g(P)$ to reduce the singularity order, such as taking $g(P) = 1-\eta^2$ for the boundary integral equation including factor $1/r^2$ so that the singular integration can be calculated analytically.

(2) The method presented here is expected to produce coefficient matrices more efficiently with higher precision in the numerical results for fewer boundary elements than the general BEM. This is especially true for the meshes of unequal length segments, a point which has been discussed less in previous research work on BEM.

(3) The approach is applicable to various boundary integral equations, such as the thin plate bending with a singularity of $1/r^2$. It is easily deduced that double integration [see eqn (3)] can be transformed into linear integration in general cases. As the integration for singular elements can be solved analytically, computation of the coefficient matrices is computationally efficient.

(4) In the present paper, the authors have given the coefficient C_{ij} in this approach (see Section 4) for the linear interpolation at boundary segments including non-smooth boundary elements.

(5) We have developed a computer program for the 2D plane elasticity, that includes the evaluation scheme discussed in this paper such as the transformation of double integration into linear integration which means no loss of CPU time. The method used in this paper can be applied to fracture, elasto-plastic mechanics problems amongst others.

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